a contradiction. If $b \equiv 1 \pmod 4$, $b \neq 1$, then choosing $x = \frac{b^2-b}{4}$ gives (1), which is impossible. As a consequence, $f(-1) = f(1) = 1$ and $f(x) = 0$ for $x \neq \pm 1$. By $P(2,1)$, we have $0 = 2f(0) + f(3) = f(1) = 1$, which is the desired contradiction.

OC224. Let $n > 1$ be an integer. An $n \times n$-square is divided into n^2 unit squares. Of these unit squares, n are coloured green and n are coloured blue, and all remaining ones are coloured white. Are there more such colourings for which there is exactly one green square in each row and exactly one blue square in each column; or colourings for which there is exactly one green square and exactly one blue square in each row?

Originally problem 5 of the 2014 South Africa National Olympiad.

We received 2 correct submissions. We present the solution by Kathleen Lewis.

There are more colourings with one green and one blue in each row. To see this, think of first placing one green square in each row; for both methods there are n^n ways to do that. If we want to place a blue square in each row, there would be $(n-1)^n$ to accomplish this, since each row has one square already coloured green. But if we wish to put a blue square in each column, the number of possibilities depends on the arrangement already made of the green squares. Suppose that there are a_i blank squares in column i. Then the number of possible arrangements of the blue squares is $\prod_{i=1}^n a_i$. The total number of available squares is $n^2 - n = n(n-1)$, so $\sum_{i=1}^n a_i = n(n-1)$. But for variables with a fixed sum, the product is greatest when all the factors are equal. So, the maximum value of $\prod_{i=1}^n a_i$ occurs when $a_1 = a_2 = \cdots = a_n = n-1$ and $\prod_{i=1}^n a_i = (n-1)^n$. In other cases, the product would be smaller, even as small as zero if the green squares were all placed in the same column. So the number of ways of placing a blue square in each column is always less than or equal to the number of ways to place the blue squares with one in each row.

OC225. Find the maximum value of real number k such that

$$\frac{a}{1+9bc+k(b-c)^2} + \frac{b}{1+9ca+k(c-a)^2} + \frac{c}{1+9ab+k(a-b)^2} \geq \frac{1}{2}$$

holds for all non-negative real numbers $a,\ b,\ c$ satisfying $a+b+c=1$.

Originally problem 5 of the 2014 Japan Mathematical Olympiad.

We received 3 correct submissions. We present the solution by Arkady Alt.

Let k be such that the original inequality holds for any non-negative real numbers a, b, c satisfying $a+b+c=1$. Then, in particular, if $a=0$ and $b=c=1/2$, we get

$$\frac{1/2}{1+k(1/2)^2} + \frac{1/2}{1+k(1/2)^2} \geq \frac{1}{2} \quad \Longleftrightarrow \quad \frac{4}{k+4} \geq \frac{1}{2} \quad \Longleftrightarrow \quad k \leq 4.$$

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Let $k \leq 4$. By Cauchy's Inequality

$$\begin{aligned}
\sum_{cyc} \frac{a}{1+9bc+k(b-c)^2} &= \sum_{cyc} \frac{a^2}{a\left(1+9bc+k(b-c)^2\right)} \\
&\geq \frac{(a+b+c)^2}{\sum\limits_{cyc} a\left(1+9bc+k(b-c)^2\right)} \\
&= \frac{1}{\sum\limits_{cyc} a\left(1+9bc+k(b-c)^2\right)} \\
&= \frac{1}{1+9abc\,(3-k)+k(ab+bc+ca)} \\
&= \frac{1}{1+9q\,(3-k)+kp},
\end{aligned}$$

where $p := ab+bc+ca$ and $q := abc$. We have

$$\begin{aligned}
&p = ab+bc+ca \leq \frac{(a+b+c)^2}{3} = 1/3, \\
&9q = 9abc \leq (ab+bc+ca)\,(a+b+c) = p, \\
&9q \geq 4p-1.
\end{aligned}$$

(Schur's Inequality $\sum\limits_{cyc} a\,(a-b)\,(a-c) \geq 0$ in p, q notation with normalization by $a+b+c=1$).

If $k \leq 3$, then

$$9q\,(3-k)+kp \leq p\,(3-k)+kp = 3p \leq 3 \cdot \frac{1}{3} = 1.$$

If $3 < k \leq 4$, then

$$9q\,(3-k)+kp \leq (4p-1)\,(3-k)+kp = k+3p\,(4-k)-3 \leq k+3\cdot\frac{1}{3}\,(4-k)-3 = 1.$$

Thus,

$$\sum_{cyc} \frac{a}{1+9bc+k(b-c)^2} \geq \frac{1}{1+9q\,(3-k)+kp} \geq \frac{1}{1+1} = \frac{1}{2}$$

for any $k \leq 4$ and, therefore, the maximum value of k is 4.